Problem 4.36

An electron is at rest in an oscillating magnetic field

$$\mathbf{B} = B_0 \cos(\omega t)\,\hat{k},$$

where B_0 and ω are constants.

(a) Construct the Hamiltonian matrix for this system.

(b) The electron starts out (at $t = 0$) in the spin-up state with respect to the x axis (that is: $\chi(0) = \chi_+^{(x)}$). Determine $\chi(t)$ at any subsequent time. *Beware:* This is a time-*dependent* Hamiltonian, so you cannot get $\chi(t)$ in the usual way from stationary states. Fortunately, in this case you can solve the time-dependent Schrödinger equation (Equation 4.162) directly.

(c) Find the probability of getting $-\hbar/2$, if you measure S_x. *Answer:*

$$\sin^2\left(\frac{\gamma B_0}{2\omega}\sin(\omega t)\right).$$

(d) What is the minimum field (B_0) required to force a complete flip in S_x?

Solution

Part (a)

The Hamiltonian matrix for an electron, which has spin 1/2, at rest in a magnetic field is given in Equation 4.158 on page 172.

$$\mathsf{H} = -\gamma\,\mathbf{B}\cdot\mathsf{S} \tag{4.158}$$

$$= -\gamma(B_0\cos\omega t\,\hat{\mathbf{z}})\cdot\mathsf{S}$$

$$= -\gamma B_0\cos\omega t\,\mathsf{S}_z$$

$$= -\gamma B_0\cos\omega t\,\frac{\hbar}{2}\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}$$

$$= \begin{bmatrix}-\frac{\gamma B_0\hbar}{2}\cos\omega t & 0\\ 0 & \frac{\gamma B_0\hbar}{2}\cos\omega t\end{bmatrix}$$

Part (b)

The governing equation for the time evolution of χ, the spin state (or spinor), is the Schrödinger equation. The spinor that starts out in the spin-up state with respect to the x-axis satisfies the following initial value problem.

$$i\hbar\frac{\partial\chi}{\partial t} = \mathsf{H}\chi, \quad t > 0$$

$$\chi(0) = \chi_+^{(x)} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

Substitute the Hamiltonian matrix.

$$i\hbar\frac{d\chi}{dt} = \begin{bmatrix} -\frac{\gamma B_0\hbar}{2}\cos\omega t & 0 \\ 0 & \frac{\gamma B_0\hbar}{2}\cos\omega t \end{bmatrix}\chi$$

$$\frac{d\chi}{dt} = \frac{1}{i\hbar}\begin{bmatrix} -\frac{\gamma B_0\hbar}{2}\cos\omega t & 0 \\ 0 & \frac{\gamma B_0\hbar}{2}\cos\omega t \end{bmatrix}\chi$$

$$\frac{d}{dt}\begin{bmatrix} \chi_1(t) \\ \chi_2(t) \end{bmatrix} = \begin{bmatrix} \frac{i\gamma B_0}{2}\cos\omega t & 0 \\ 0 & -\frac{i\gamma B_0}{2}\cos\omega t \end{bmatrix}\begin{bmatrix} \chi_1(t) \\ \chi_2(t) \end{bmatrix}$$

$$\begin{bmatrix} \chi_1'(t) \\ \chi_2'(t) \end{bmatrix} = \begin{bmatrix} \frac{i\gamma B_0}{2}\cos\omega t\,\chi_1(t) \\ -\frac{i\gamma B_0}{2}\cos\omega t\,\chi_2(t) \end{bmatrix}$$

This matrix equation leads to a decoupled system of first-order ODEs. Solve this system for χ_1 and χ_2.

$$\begin{cases} \chi_1'(t) = \frac{i\gamma B_0}{2}\cos\omega t\,\chi_1(t) \\ \chi_2'(t) = -\frac{i\gamma B_0}{2}\cos\omega t\,\chi_2(t) \end{cases}$$

$$\begin{cases} \frac{\chi_1'}{\chi_1} = \frac{i\gamma B_0}{2}\cos\omega t \\ \frac{\chi_2'}{\chi_2} = -\frac{i\gamma B_0}{2}\cos\omega t \end{cases}$$

Rewrite each left side by using the chain rule and then integrate both sides with respect to t.

$$\begin{cases} \dfrac{d}{dt}\ln\chi_1 = \dfrac{i\gamma B_0}{2}\cos\omega t \\[2ex] \dfrac{d}{dt}\ln\chi_2 = -\dfrac{i\gamma B_0}{2}\cos\omega t \end{cases}$$

$$\begin{cases} \ln\chi_1 = \dfrac{i\gamma B_0}{2}\left(\dfrac{\sin\omega t}{\omega}\right) + C_1 \\[2ex] \ln\chi_2 = -\dfrac{i\gamma B_0}{2}\left(\dfrac{\sin\omega t}{\omega}\right) + C_2 \end{cases}$$

$$\begin{cases} \chi_1(t) = \exp\left(\dfrac{i\gamma B_0}{2\omega}\sin\omega t + C_1\right) \\[2ex] \chi_2(t) = \exp\left(-\dfrac{i\gamma B_0}{2\omega}\sin\omega t + C_2\right) \end{cases}$$

$$\begin{cases} \chi_1(t) = A_1\exp\left(\dfrac{i\gamma B_0}{2\omega}\sin\omega t\right) \\[2ex] \chi_2(t) = A_2\exp\left(-\dfrac{i\gamma B_0}{2\omega}\sin\omega t\right) \end{cases}$$

The general solution for χ is then

$$\chi(t) = \begin{bmatrix} \chi_1(t) \\ \chi_2(t) \end{bmatrix} = \begin{bmatrix} A_1\exp\left(\frac{i\gamma B_0}{2\omega}\sin\omega t\right) \\[1ex] A_2\exp\left(-\frac{i\gamma B_0}{2\omega}\sin\omega t\right) \end{bmatrix}.$$

Use the initial condition to determine A_1 and A_2.

$$\chi(0) = \begin{bmatrix} A_1\exp(0) \\[1ex] A_2\exp(0) \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\[1ex] \frac{1}{\sqrt{2}} \end{bmatrix} \quad \Rightarrow \quad \begin{cases} A_1 = \dfrac{1}{\sqrt{2}} \\[2ex] A_2 = \dfrac{1}{\sqrt{2}} \end{cases}.$$

Therefore, the spin state at any subsequent time t is

$$\chi(t) = \begin{bmatrix} \frac{1}{\sqrt{2}}\exp\left(\frac{i\gamma B_0}{2\omega}\sin\omega t\right) \\[1ex] \frac{1}{\sqrt{2}}\exp\left(-\frac{i\gamma B_0}{2\omega}\sin\omega t\right) \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} \exp\left(\frac{i\gamma B_0}{2\omega}\sin\omega t\right) \\[1ex] \exp\left(-\frac{i\gamma B_0}{2\omega}\sin\omega t\right) \end{bmatrix}.$$

Part (c)

The probability of measuring $-\hbar/2$ for the component of spin angular momentum along the x-direction is the modulus squared of the component of $|\chi\rangle$ along $|\chi_-^{(x)}\rangle$.

$$P\left(-\frac{\hbar}{2}\right) = \left|c_-^{(x)}\right|^2$$

$$= \left|\langle\chi_-^{(x)}\,|\,\chi\rangle\right|^2$$

$$= \left|\chi_-^{(x)\dagger}\chi\right|^2$$

The normalized eigenfunction (or rather eigenspinor in this context) associated with the eigenvalue $-\hbar/2$ for S_x is in Equation 4.151 on page 169.

$$\chi_-^{(x)} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Consequently,

$$P\left(-\frac{\hbar}{2}\right) = \left|\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}^\dagger \frac{1}{\sqrt{2}}\begin{bmatrix} \exp\left(\frac{i\gamma B_0}{2\omega}\sin\omega t\right) \\ \exp\left(-\frac{i\gamma B_0}{2\omega}\sin\omega t\right) \end{bmatrix}\right|^2$$

$$= \frac{1}{4}\left|\begin{bmatrix} 1 & -1 \end{bmatrix}\begin{bmatrix} \exp\left(\frac{i\gamma B_0}{2\omega}\sin\omega t\right) \\ \exp\left(-\frac{i\gamma B_0}{2\omega}\sin\omega t\right) \end{bmatrix}\right|^2$$

$$= \frac{1}{4}\left|\exp\left(\frac{i\gamma B_0}{2\omega}\sin\omega t\right) - \exp\left(-\frac{i\gamma B_0}{2\omega}\sin\omega t\right)\right|^2$$

$$= \frac{1}{4}\left[\exp\left(\frac{i\gamma B_0}{2\omega}\sin\omega t\right) - \exp\left(-\frac{i\gamma B_0}{2\omega}\sin\omega t\right)\right]\left[\exp\left(\frac{i\gamma B_0}{2\omega}\sin\omega t\right) - \exp\left(-\frac{i\gamma B_0}{2\omega}\sin\omega t\right)\right]^*$$

$$= \frac{1}{4}\left[\exp\left(\frac{i\gamma B_0}{2\omega}\sin\omega t\right) - \exp\left(-\frac{i\gamma B_0}{2\omega}\sin\omega t\right)\right]\left[\exp\left(-\frac{i\gamma B_0}{2\omega}\sin\omega t\right) - \exp\left(\frac{i\gamma B_0}{2\omega}\sin\omega t\right)\right]$$

$$= \frac{1}{4}\left[1 - \exp\left(\frac{i\gamma B_0}{\omega}\sin\omega t\right) - \exp\left(-\frac{i\gamma B_0}{\omega}\sin\omega t\right) + 1\right]$$

$$= \frac{1}{2}\left[1 - \frac{\exp\left(\frac{i\gamma B_0}{\omega}\sin\omega t\right) + \exp\left(-\frac{i\gamma B_0}{\omega}\sin\omega t\right)}{2}\right].$$

Therefore,

$$P\left(-\frac{\hbar}{2}\right) = \frac{1}{2}\left[1 - \cos\left(\frac{\gamma B_0}{\omega}\sin\omega t\right)\right]$$

$$= \frac{1}{2}\left\{1 - \cos\left[2\left(\frac{\gamma B_0}{2\omega}\sin\omega t\right)\right]\right\}$$

$$= \frac{1}{2}\left\{1 - \left[1 - 2\sin^2\left(\frac{\gamma B_0}{2\omega}\sin\omega t\right)\right]\right\}$$

$$= \frac{1}{2}\left[2\sin^2\left(\frac{\gamma B_0}{2\omega}\sin\omega t\right)\right]$$

$$= \sin^2\left(\frac{\gamma B_0}{2\omega}\sin\omega t\right).$$

Part (d)

The spin state starts out as spin-up with respect to the x-axis. This means a measurement of S_x would result in $+\hbar/2$ with certainty. For a complete flip in S_x to happen, a measurement of S_x would have to yield $-\hbar/2$ with certainty, that is,

$$P\left(-\frac{\hbar}{2}\right) = 1 \quad \Rightarrow \quad \sin^2\left(\frac{\gamma B_0}{2\omega}\sin\omega t\right) = 1$$

$$\frac{\gamma B_0}{2\omega}\sin\omega t = \frac{\pi}{2}$$

$$B_0 = \frac{\pi\omega}{\gamma\sin\omega t}.$$

The minimum value of B_0 occurs when $\sin\omega t = 1$.

$$B_0 = \frac{\pi\omega}{\gamma}$$